Flow Localization in Non-Linear Random Networks

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Convex separable linearly-constrained optimization:

Cost objective:

\[
\arg \min_x \left\{ f(x) = \sum f_i(x_i) \right\} \quad (1)
\]

Constraints:

\[Ax = b\] \quad (2)
Convex separable network optimization:

Energy functional:

$$\arg \min_{x} \left\{ f(x) = \sum_{e(i,j)} f(i,j) \left[ x(i,j) \right] \right\}$$ \quad (3)

Flow conservation:

$$\sum_{e(i,j)} x(i,j) - \sum_{e(j,i)} x(j,i) = b_i$$ \quad (4)

\textbf{x} \hspace{0.5em} \textbf{flow} through the arcs

\textbf{f} \hspace{0.5em} \textbf{total cost} function—\textit{separable} and \textit{convex}

\textit{f}_{(i,j)} \hspace{0.5em} \textbf{elemental costs} functions—\textit{convex}

\textbf{b} \hspace{0.5em} \textbf{supplies–demands} at the nodes
Why is this kind of problem crucial to materials science?

1. Many transport phenomena in disordered systems can be modeled as network optimization problems:

   - **grain boundaries in superconducting materials** (*critical currents*)
   - **flow through porous media** (*critical pressure*)
   - **breakdown in varistors** (*threshold electric field*)
   - **river network formation** ...
In these systems, the elemental cost functions are related to the **potential–flow characteristics** $t_i(x_i)$ of the arcs:

$$f_i(x_i) = \int_{\zeta=0}^{x_i} t_i(\zeta) d\zeta$$

- **voltage–current**: $t_i(x_i) \equiv V_i(I)$

- **pressure–flow**: $t_i(x_i) \equiv P_i(Q_i)$

Since $V(I)$, $P(Q)$ are monotonically increasing, the elemental costs are convex!
Examples of Potential-Flow Characteristics

Power-law costs \( f_i(x_i) = R_i |x_i|^\alpha \)

By changing \( \alpha \) we cover a whole range of behaviour:

- \( \alpha > 1 \) strictly convex (flow has cycles)

- \( 0 < \alpha \leq 1 \) concave (spanning tree-like flows)
\[ f_i(x_i) = R_i |x_i|^\alpha \]  \hspace{1cm} (5)

Limiting cases:

- $\alpha = 0$—*Shortest paths* (Dijkstra tree—combinatorial)

- $0 < \alpha < 1$—*River networks* (Simulated annealing)

- $\alpha = 1$—*Linear programming* (Simplex network optimization)

- $\alpha > 1$—*Resistive networks* (Convex network optimization)

- $\alpha \to \infty$—*Maximum Flow* (Min-Cut Surface—combinatorial)
Cost functions with thresholds

These exhibit **macroscopic flow localization**

- tanh-like (*critical flow* $x_i^{(0)}$)—related to Max-Flow (Min-Cut)

$$
t_i(x_i) = R_i \left| x_i \right|^{(1+\beta)} \left[ 1 + \tanh \left( \frac{\left| x_i \right| - x_i^{(0)}}{\zeta} \right) \right]
$$

*Flow saturation occurs on surfaces*

- tanh\(^{-1}\)-like (*critical potential* $t_i^{(0)}$)—related to Min-Path:

$$
x_i(t_i) = R_i \left| t_i \right|^{(1+\beta)} \left[ 1 + \tanh \left( \frac{\left| t_i \right| - t_i^{(0)}}{\zeta} \right) \right]
$$

*Flow onset occurs on paths*
Large Scale Convex Network Optimization (LSCNO) Library

Problems:

1. People that have worked on network optimization algorithms (computer scientists, applied mathematicians, engineers, operations research) don’t care about the needs of materials science physicists:
   Very large, near-neighbour, i.e fully separable, unbounded, non-integer, convex (or strictly convex)

2. The only public-domain software library for large-scale sparse non-linear optimization is the LSNNO library—better (dual-based, interior-point) algorithms have been developed in the mean time
Solutions:

1. We are physicists and we want to share our work with others

2. We are writing our own highly efficient, parallel **LSCNO library**:
   
   - It will be written in *Fortran 90–High Performance Fortran*
   
   - It will have both *serial* and *parallel* versions
   
   - Considerable progress has been made, but a *lot is to be done.*
Others can help:

- *Applied mathematicians* (algorithm tuning)

- *Computer scientists* (parallel implementation)

- *Physicists* (testing and feedback)

Visit

http://computation.pa.msu.edu/NO/ for details and codes!